MATH 2040A Lecture 4 (Sep 19, 2016)

§ Change of basis (textbook §2.5)

Recall: $V \supseteq \beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ $\dim V = n$

$W \supseteq \gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$ $\dim W = m$

$$\mathcal{L}(V,W) = \left\{ \begin{array}{c} T: V \to W \\ \text{linear} \end{array} \right\} \underset{\beta,\gamma}{\overset{\text{"}\cong\text{"}}{\longleftrightarrow}} \left\{ \begin{array}{c} m \times n \text{ matrices} \\ A \text{ over } \mathbb{F} \end{array} \right\} = M_{m\times n}(\mathbb{F})$$

Remember: $V \cong_\beta \mathbb{F}^n$ $W \cong_\gamma \mathbb{F}^m$

$$[T]_\beta^\gamma = \begin{pmatrix} | & & | \\ [T\vec{v}_1]_\gamma & \cdots\cdots & [T\vec{v}_n] \\ | & & | \end{pmatrix} \in M_{m\times n}(\mathbb{F})$$

A Quick E.g. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$

proj. onto x-axis : $T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$

(1) Take $\beta = \{\vec{e}_1, \vec{e}_2\}$ ← std basis for $\mathbb{R}^2$

$$T(\vec{e}_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \vec{e}_1 + 0 \cdot \vec{e}_2$$

$$T(\vec{e}_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \vec{e}_1 + 0 \cdot \vec{e}_2$$

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{aligned} tr &= 1 + 0 = 1 \\ det &= 0 \end{aligned}$$

(2) Take $\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$[T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \begin{aligned} tr &= \frac{1}{2} + \frac{1}{2} = 1 \\ det &= \frac{1}{4} - \frac{1}{4} = 0 \end{aligned}$$

Def$^{\underline{n}}$: $A, B \in M_{n\times n}(\mathbb{F})$ similar (ie. $A \sim B$)

if $\exists$ invertible $Q \in M_{n\times n}(\mathbb{F})$ s.t.

$$\boxed{A = Q^{-1}BQ}$$

FACTS: (a) "$\sim$" is an equiv. relation

(b) If $A \sim B$, then

$$\boxed{\begin{aligned} tr\,A &= tr\,B \\ det\,A &= det\,B \end{aligned}}$$

Proof: (a) Ex. (b) $tr(AB) = tr(BA)$

$det(AB) = det(BA)$ $(= det\,A \cdot det\,B)$

$\square$

Q: Why define "$A \sim B$" like this?

Theorem (Change of coord. formula)

Given $T: V \to V$ (dim $V = n$), suppose $\beta, \beta' \subseteq V$ basis.

$$\boxed{[T]_{\beta'} = Q^{-1}[T]_\beta Q}$$

where $Q = [I_V]_{\beta'}^{\beta}$ = change of coord. matrix

From Previous E.g. ....

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = Q^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q$$

To find this:

$Q = [I_V]_{\beta'}^{\beta}$

$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ $\because \beta$ std.

$$\begin{cases} I_V\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ I_V\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases} \Rightarrow Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Properties of Q: (1) Q invertible.

$Q : \beta' \to \beta$ (2) $\forall \vec{v} \in V$, $[\vec{v}]_\beta = Q[\vec{v}]_{\beta'}$

$Q^{-1} : \beta \to \beta'$

Proof: (1) Consider

$$\underset{\beta}{V} \xrightarrow{I_V} \underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V}$$

$$I_V = I_V \circ I_V$$

$$\Rightarrow \quad I = [I_V]_\beta = [I_V \circ I_V]_\beta^\beta = \underset{\overset{\shortparallel}{Q}}{[I_V]_{\beta'}^{\beta}} \; \underset{\overset{\shortparallel}{Q^{-1}}}{[I_V]_{\beta}^{\beta'}}$$

$\square$

(2) Consider $I_V : \underset{\beta'}{V} \to \underset{\beta}{V}$

$$\Rightarrow \quad [\vec{v}]_\beta = [I_V(\vec{v})]_\beta$$

$$= \underset{\overset{\shortparallel}{Q^{-1}}}{[I_V]_{\beta}^{\beta'}} [\vec{v}]_{\beta'}$$

$\square$

Proof of Change of coord. formula:

Consider

$$V \xrightarrow{I_V} V \xrightarrow{T} V \xrightarrow{I_V} V$$
$$\beta' \qquad \beta \qquad \beta \qquad \beta'$$

$$\Rightarrow [T]_{\beta'} = [I_V \circ T \circ I_V]_{\beta'}$$

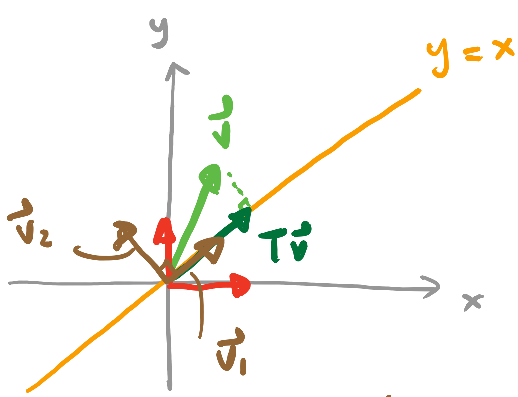
$$= [I_V]_{\beta}^{\beta'} [T]_{\beta}^{\beta} [I_V]_{\beta'}^{\beta}$$

$$= Q^{-1} [T]_{\beta} Q$$

$\square$

Q: Why care about this formula?

Eg.: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ "proj. onto $y=x$".



Q: Find $[T]_{\beta}$, β = std basis

If pick $\beta' = \{\vec{v}_1, \vec{v}_2\}$

$$[T]_{\beta'} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{std. proj. matrix}}$$

$$Q = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} \qquad [T]_{\beta} = Q [T]_{\beta'} Q^{-1}$$

§ Eigenvalues / Eigenvectors

Q: Given $T: V \to V$, $\exists$? a good basis $\beta \subseteq V$

s.t. $[T]_\beta$ is "simple" ?

Note: "simple" $\longleftrightarrow$ "diagonal".

e.g. $O = \begin{pmatrix} 0 & 0 & \cdots \\ 0 & & \\ \vdots & & 0 \\ 0 & & \end{pmatrix}$, $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$ ← "diagonal matrices"

why? **Remember**: $AB \neq BA$ in general.

But: $D_1 D_2 = D_2 D_1$ if D_1, D_2 diagonal.

(Pf: Exercise)

E.g.: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 & 0 \\ 0 & 2 \cdot 4 \end{pmatrix}$

If $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ then $\det D = d_1 d_2 \cdots d_n$

diagonal

(or triangular, ie $D = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ or $\begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$)

Def$^{\underline{n}}$: $T: V \to V$ linear is diagonalizable

iff $\exists\ \beta \subseteq V$ basis s.t.

$[T]_\gamma = A$

$$[T]_\beta = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \text{ "diagonal"}$$

Remark: $A \in M_{n\times n}(\mathbb{F})$ diagonalizable

iff $A \sim D$, a diagonal matrix.

i.e. $\exists\ Q$ invertible s.t. $Q^{-1}AQ = D$ ← diagonal.

Look at it more carefully ......

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ s.t.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \iff \boxed{T\vec{v}_i = \lambda_i \vec{v}_i}$$

eigenvectors ($\vec{v}_i$), eigenvalues (λ_i)

for $i = 1, \dots, n$

$$\| $$

$$\begin{pmatrix} | & & | \\ [T\vec{v}_1]_\beta & \cdots & [T\vec{v}_n]_\beta \\ | & & | \end{pmatrix}$$

Def^{n}: Let $T: V \to V$. A $\vec{v} \in V$ is an eigenvector of T if ① $\boxed{\vec{v} \neq \vec{0}}$

② $\boxed{T\vec{v} = \lambda \vec{v}}$ for some $\lambda \in \mathbb{F}$

eigenvalue asso. with $\vec{v}$

Remark: $A \in M_{n \times n}(\mathbb{F})$.

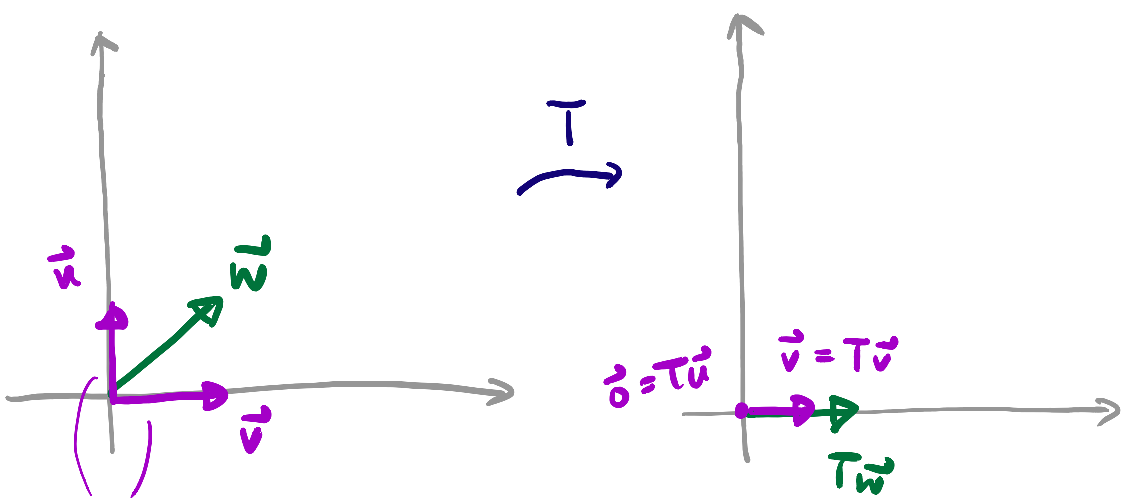
$$\boxed{A\vec{v} = \lambda \vec{v}}$$

eigenvector

eigenvalue

Geometric meaning of eigenvectors / eigenvalues:

Consider T = proj. onto x-axis : $\mathbb{R}^2 \to \mathbb{R}^2$



eigenvectors of T ∵ direction remains unchanged

(magnitude may change)

E.g. $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvector w/. $\lambda = 1$ (ie $T\vec{e}_1 = \vec{e}_1$)

$\underset{\vec{0}}{\underset{\#}{\vec{e}_2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvector w/. $\underbrace{\lambda = 0}_{\overset{\|}{0}}$ (ie $T\vec{e}_2 = \vec{0}$)

<u>Two facts</u>:

(1) $\vec{v} \in V$ eigenvector of T w/. $\lambda = 0$

$\iff \vec{0} \neq \vec{v} \in N(T) := \{ \vec{x} \in V : T\vec{x} = \vec{0} \}$

(2) $\vec{v} \in V$ eigenvector of T

$\iff a \cdot \vec{v} \in V$, $a \neq 0$, are eigenvectors of T

(with same λ)

<u>Why</u>? If $T\vec{v} = \lambda \vec{v}$, then

$$T(a \cdot \vec{v}) = a T(\vec{v}) = a(\lambda \vec{v})$$
$$= \lambda (a \vec{v})$$

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